Quadratic Programming Applied to Modern Portfolio Selection

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Abstract

The mean-variance formulation by Markowitz in 1956 and its efficient solution by Wolfe in 1959 paved a foundation for modern portfolio selection. In this work we start reviewing basic concepts about portfolio selection, showing one starting solution and then the mean-variance analysis proposed by Markowitz. We show an algorithm for efficient frontier derivation, proposed by Wolfe, and analyze the performance of 24 portfolios generated by our implementation of this method, 12 of then in a bull market and the other 12 in a bear market. After that we finish our work presenting some limitations of this formulation and recent studies where preference for skewness is introduced. Another nonlinear models are also suggested.

Key Words: Quadratic Programming, Portfolio Selection, Utility Function, Skewness.

1 Basic Definitions

1.1 The Problem

Consider an investor that seeks a best allocation of wealth among a basket of risky assets, called portfolio. The best can be defined as an allocation such that the risk incurred is minimum for that level of expected return or the expected return is maximum for that level of risk.
The data of the problem consists in an array of returns, where each component *i* of this array is the expected return to the asset *i* in the considered horizon:

$$r = (r_1, r_2, ..., r_n)$$

We have also a covariance matrix as the shown below:

$$Cov = \begin{bmatrix} \sigma_1^2 & \sigma_{21} & \cdots & \sigma_{n1} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{bmatrix} \qquad \begin{array}{l} \sigma_i^2 = VARIANCE(i) \\ \sigma_{ij} = COVARIANCE(i, j) \end{array}$$

This matrix is positive semidefinite.
The expected return for a portfolio with *n* assets is *x'r*, where each component *i* of the array $x = (x_1, x_2, ..., x_n)$ is the fraction of the investor wealth allocated in the asset *i*. The portfolio risk is *x´Cov x*. By assumption, the investor will allocate all his wealth in the selected portfolio.
As he wants to obtain the optimal relation between return and risk, the nonlinear programming problem needed to solve is the following:

1.1.1

$$\max \quad \frac{x'r - R_f}{\sqrt{x'Cov.x}} \qquad s.t. \quad \sum_{i=1}^{n} x_i = 1$$

Where R_f is the return of a riskless asset.
There are many methods of nonlinear programming that may be used to solve this problem. One of them is the following. Let's substitute the constraint in the objective function and solve an uncostrained problem. Of course, this does not work in every maximization problem. It works here because the differential equations of the problem are homogenous of degree zero.
Let be $R_f = 1R_f$. Now the problem can be stated as the following:

1.1.2

$$\max \quad \theta = \frac{\sum_{i=1}^{n} x_i (r_i - R_f)}{\left[\sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq 1}}^{n} x_i x_j \sigma_{ij} \right]^{1/2}}$$

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To solve it, we have only to find the gradient of the objective function. In the point where this array is equal to zero, we'll have a maximum as this new problem is unconstrained and the second derivative of objective function is always negative. This fact is guaranteed by structure of the problem, as proved in [1].

So this is equivalent to solve the following system of linear equations:

1.1.3 $$\frac{d\theta}{dx_i} = 0, \forall i$$

But is proved in [1] that, for each *i*, the equation above is equivalent the equation:

$$-\left(wx_i\sigma_{1i} + wx_2\sigma_{2i} + ... + wx_i\sigma_i^2 + ... + wx_{n-1}\sigma_{n-1i} + wx_n\sigma_{ni}\right) + r_i - R_f = 0$$

with $$w = \frac{\sum_{i=1}^{n} x_i(r_i - R_f)}{\sum_{i=1}^{n} x_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} x_i x_j \sigma_{ij}}$$

Therefore solve the proposed system of differencial equations is reduced to solve the system $Cov.Z = \overline{R}$ where Z is an array such that the component $Z_k = wx_k$ and $\overline{R}$ is an array such that $\overline{R_k} = r_k - R_f$ for all *k*.

The last step of this basic problem is to solve the system finding *Z* and then multiplying the solution found by a factor such that the sum of the components of *x* is equal to 1, as *Z* is proportional to *x*.

1.2 Extending the Problem

The proposed problem may ser extended to become more realistic. There are several possible extensions:

- Add cash constraints: $l \le x \le u$;
- Add many linear constraints as: book value concentration, portfolio's beta, liquidity of the assets included in the portfolio, dividend yield, and more;
- Parameterize the risk aversion of the investor. In this way, the ideal algorithm would find more than the portfolio with the maximum expected return possible per risk unit. It would find for all the feasible returns, the portfolio that has this expected return with the minimum risk possible. Obtaining this portfolios we will have the Markowitz Efficient Frontier that is, by definition, the set of portfolios with minimum risk for all the feasible returns.

Therefore the extended problem is stated as the following:

1.2.1 $$\begin{aligned} \min\ & x' \, Cov \, x - \lambda x' R \\ s.t.\ & Ax = b \\ & Cx \le d \\ & l \le x \le u \end{aligned}$$

We want to find all the feasible solutions of this problem, i.e. for all the values of $\lambda \ge 0$. Doing this, we'll be solving a quadratic parametric problem, whose algorithm was initally proposed by Harry Markowitz in 1956 and complemented by Wolfe in 1959. This algoritm has received the name of Wolfe's method. We will now talk more about his details and implementation.

2 Methods to Solve Quadratic Problems

2.1. Formulation of the Quadratic Model

The problem modeled in **1.2.1** may be reduced to the following quadratic programming problem (QP):

2.1.1 $$\begin{aligned} \min\ & x'Cx + \lambda x' r \\ s.a.\ & x \in \Im, \Im = \{x \mid Ax \le b, x \ge 0\} \end{aligned}$$

where *C* is a matrix *n* by *n* such that, by assumption:

- *C* is positive semidefinite
- There is some $q \in \Re^n$ such that $r = Cq$.

To adapt (QP) to **1.2.1**, *r* now is an array such that each component *i* is the return of the asset *i* subtracted the return of a riskless asset. The equality constraints of **1.2.1** may be substituted by two inequality constraints and if we want to allow short sales we can introduce artificial variables. Since the objective function is quadratic and convex, to solve (QP) we have to find a feasible solution that meets the Kuhn-Tucker conditions. In this case, they are necessary and sufficient optimality criterion. A Simplex-based method that solve this problem was proposd by Wolfe in [2], with two variations. One solve the problem for a fixed value of λ (called "short form" in the original paper) and the other solve the problem using λ as a parameter, finding a set of critical solutions that can generate all feasible solutions (called "long form" in the original paper). These methods solve the problem (QP) in a finite number of steps.

Markowitz had proposed another method for optimization of a quadratic function with linear constraints 3 years before of the Wolfe's proposal. Although this method is different of the Wolfe's method, one can proof that both find the same results.

2.2. The Wolfe's Method

The method shown below solve the problem for one fixed value of the parameter. The first step is to introduce non-negative slack arrays $y \in \Re_+^m, v \in \Re_+^n$ defined by:

$$y = b - Ax \geq 0,$$

$$v = Cx + A'u + \lambda r \geq 0,$$

where $u \in \Re_+^m$ is the array of Lagrange multipliers.

A pair of variables x_j, v_j will be called complementary pair and the other pair of variables y_i, v_i will also be complementry. With this notation the Kuhn-Tucker conditions of the model above take the following form:

2.2.1 (a) $x, y, v, u \geq 0$

(b) $Ax + y = b$

(c) $-Cx + v - A'u = \lambda r$

(d) $x'v + y'u = 0$

In [3] is demonstrated the following theorem:

The parts x, y *of any solution for the Kuhn-Tucker conditions are a vertex of the following polytope:*

$$\mathrm{M} = \{[x', u']\ t.q.\ Ax \leq b, -Cx - A'u \leq \lambda r, x \geq 0, u \geq 0\}$$

In this way, the problem of finding the solution of **2.2.1** is reduced to the problem of finding an admissible basis *B* of the matrix:

$$D = \begin{bmatrix} A & E & 0 & 0 \\ -C & 0 & E & -A' \end{bmatrix}$$

The basis is admissible in the sense of $B^{-1}\begin{bmatrix} b \\ \lambda r \end{bmatrix} \geq 0$.

In the matrix *D* *j*-th column of $\begin{bmatrix} A \\ -C \end{bmatrix}$ (the variable x_j) and the *j*-th column of $\begin{bmatrix} 0 \\ E \end{bmatrix}$ (the variable v_j) must not be basic at the same time, and the same can be applied to $\begin{bmatrix} E \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -A' \end{bmatrix}$ associated to the variables y_i and u_i.

For this purpose we introduce artificial variables in **2.2.1(b)** and perform the Phase I of the Simplex method until we form an initial basis for **2.2.1(b)**. After that, we introduce a new set of artificial variables in **2.2.1(c)** to create an initial artificial basis without the columns $\begin{bmatrix} 0 \\ E \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -A' \end{bmatrix}$, i.e. without any v or y component.

So **2.2.1(d)** is satisfied at the beginning with $v = 0, u = 0$. The complementarity property $x'v + y'u = 0$ will be preserved while the basic artificial variables are eliminated. A complementary variable pair x_j, v_j or y_i, u_i will never both be basic. In this way, the solution to **2.2.1** will be obtained when the *x* part solves **2.1.1.**

After the brief description given above, we can formulate a Quadratic Programming algoritm as the described below:

Phase 0. Solve the following linear programming problem:

2.2.2 max $-e'z_1$

s.t. $x^* \geq 0$

$Dx^* = b^*$

with $D \equiv [\, A^* \quad E^* \quad E \,]$, $x^* \equiv \begin{bmatrix} x \\ y \\ z_1 \end{bmatrix}$, A^*, E^* and b^* obtained in the following way:

For each line *j* of the matrix A^*,

$$A_j^* = \begin{cases} A_j, & se \quad b_j \geq 0 \\ -A_j, & se \quad b_j < 0 \end{cases};$$

For each line *j* of the matrix E^*,

$$E_j^* = \begin{cases} E_j, & se \quad b_j \geq 0 \\ -E_j, & se \quad b_j < 0 \end{cases};$$

And for each *j*, $b_j^* = |b_j|$. *E* is the identity matrix.

Having initial feasible solution $x^* \equiv \begin{bmatrix} 0 \\ 0 \\ b^* \end{bmatrix}$ and initial basis $D_{ini} \equiv [\, E \,]$.

Case 1. Let be $\overline{x}$, $\overline{y}$ and $\overline{z_1}$ the arrays found in the final solution. If $-e'\overline{z_1} \neq 0$ then stop – the problem is unfeasible. Else, go to phase 1.

Phase 1. Solve the following linear programming problem:

2.2.3 max $-e'z_2$

s.t. $x^{f1} \geq 0$

$D^{f1}x^{f1} = b^{f1}$

With $$D^{f1} \equiv \begin{bmatrix} A^* & E^* & 0 & 0 & 0 \\ -C & 0 & E & -A' & E^{f1} \end{bmatrix},$$

$$x^{f1} = \begin{bmatrix} x \\ y \\ v \\ u \\ z_2 \end{bmatrix}, \; b^{f1} = \begin{bmatrix} b^* \\ \lambda r \end{bmatrix}$$ and also

$$E_j^{f1} = \begin{cases} E_j, & if \quad (C\bar{x}+\lambda r)_j \geq 0 \\ -E_j, & if \quad (C\bar{x}+\lambda r)_j < 0 \end{cases},$$ for each line j of the matrix E^{f1}.

Let be D_{fim} the final phase 0 basis and C_{fim} the matrix formed by columns of $[-C \quad 0]$ corresponding to the columns of $[A^* \quad E^* \quad E]$ selected by D_{fim}. To solve the phase 1 problem, take as initial basis the matrix:

$$B^1 = \begin{bmatrix} D_{fim} & 0 \\ C_{fim} & E^{f1} \end{bmatrix}$$

whose initial feasible solution will be:

2.2.4 $$\begin{bmatrix} \hat{x}^1 = \bar{x} \\ \hat{y}^1 = \bar{y} \\ \hat{v}^1 = 0 \\ \hat{u}^1 = 0 \\ \hat{z}_2^1 = E^{f1}(Cx + \lambda r) \end{bmatrix}$$

Solve the problem **2.2.3** by the Simplex method, starting with the inicial basis B^1 and its respective basic solution **2.2.4** apllying the following entry basis restriction:

2.2.5 If $\left.\begin{matrix} x_j \\ v_j \\ y_i \\ u_i \end{matrix}\right\}$ is basic then $\left\{\begin{matrix} v_j \\ x_j \\ u_i \\ y_i \end{matrix}\right.$ must not enter the basis (note that the solution **2.2.4** satisfy the condition **2.2.5**).

Without the entry basis restriction, we can be sure that when the algorithm above stops $e'z_2 = 0$ (since the problem **2.2.3** is always feasible). Wolfe proved in [2] that even with this condition, when the algorithm stops the optimal value of **2.2.3** will be 0. He also proved the following theorem:

The problem ***2.1.1*** *has optimal solution if* $\Im$ *is not empty and the part corresponding to x of the optimal solution of* ***2.2.3*** *under the condition* ***2.2.5*** *is this solution.*

In this form, we made a more sophisticated model to the proposed problem in the part 1 and we shown a method based in the Simplex to solve it. The method shown solves the problem **2.1.1** when the parameter λ is constant in a finite number of steps. It means that, having the investor's risk aversion level (λ can represent the inverse of the investor's risk aversion), having the covariance matrix, the array of returns of each market asset and having the linear constraints, we can obtain the best portfolio concerning risk and expected return. It's known in literature as *efficient portfolio*.

Although if the our objective is obtain all the efficient portfolios, i.e. obtain all the portfolios whose risk is minimum (or expected return is maximum), we will have to solve the problem varying λ from $-\infty$ to 0. But it doesn't seem to be reasonable, since it will be very computationaly expensive (the algorithm is based in the Simplex Method and its worst case is exponential). One can simply try to choose a random set of values in the required interval and solve the problem only for them, obtaining a solution for a particular value between two actual solutions as a linear combination of them. It seems reasonable but doing this we will be subject to the risk of choose the wrong values, choosing values that do not have important information. The parametric version is like this proposal, but the values are obtained with a simple criteria.

To solve this problem, Wolfe proposed in the same article a variation of the algorithm of quadratic programming , called "long form" in the original paper, that find in a more efficient way a set of solutions that can generate a solution for any value $\lambda \leq 0$. From now on, we will study this variation.

2.3. The Parametric Version

As noted before, the optimal solution must:

- Have maximum expected return (*x'r*) to its risk level and;
- Have minimum risk (*x'Cov x*) to its expected return level.

Not only this, we want an algorithm that find all portfolios of minimum risk without varying the value or the risk aversion ($1/\lambda$) and solving the problem repeated times. Intuitively, the parametric algorithm would do the hard work just one time and after that to find any other solution would require just more one step. Wolfe proposed in its "long form" an algorithm based in sensibility analysis of the linear Kuhn-Tucker conditions. The Quadratic Parametric Problem, is transformed in a Linear Programming problem and the critical values críticos of λ are found. Each critical value of λ corresponds to one solution (called by

Markowitz as “corner portfolio”) and for any value of λ between two critical values, its corresponding solution is a convex combination the two nearest critical solutions. Now the work is to find all the critical values of λ. Having this values, draw all the Efficient Frontier becomes trivial.
The proposed algorithm solves a quadratic problem parametric of the folowing form:

2.3.1
$$\begin{aligned} &\min \quad x'Cx - \lambda x'r \\ &s.a. \quad x \in \Im, \Im = \{x \mid Ax \le b, x \ge 0\} \end{aligned}$$

Note that now the algorithm just find values of $\lambda \ge 0$.
The Parametric Quadratic Programming algorithm, known also as *Phase 2 of the Wolfe’s Method* is described below:

Step 1. Solve the problem **2.3.1** with $\lambda = 0$. Let B^0 be the final basis and $x^0, y^0, v^0, u^0, z_2^0$ the final solution. If any column corresponding to a z_2 component is in B^0 - so this vertex is degenerated and this component values 0 – continue the *Phase 1* until no component of z_2 remains in the basis.
Step 2. Form the folowing Linear Programming Problem:

2.3.2
$$\begin{aligned} &\max \quad 2\lambda \\ &s.t. \quad M\overline{\overline{x}} = \overline{\overline{b}} \\ &\qquad\ \ \overline{\overline{x}} \ge 0 \end{aligned}$$

$$M = \begin{bmatrix} A & E & 0 & 0 & 0 \\ -C & 0 & E & -A' & r \end{bmatrix}$$

2.3.3 where $\overline{\overline{x}} = \begin{bmatrix} x \\ y \\ v \\ u \\ \lambda \end{bmatrix}$

$$\overline{\overline{b}} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

with $\lambda^0 = 0, x^0, y^0, v^0, u^0$ as initial feasible solution and B^0 its corresponding basis. Solve this problem by the Simplex Phase 2 with the restriction **2.2.5**.
Step 3. If with the restriction **2.2.5** it’s impossible to continue, i.e. the optimal solution is the initial feasible solution then stop! There’s no solution to **2.3.1** with $\lambda \ge 0$.

Step 4. If $-\lambda$ can be decreased then **2.3.2** is unbounded (proved by Wolfe in [2]). So at each iteration *r* of the *s* iterations of the Simplex with the entry basis restriction **2.2.5** we obtain a solution $\overline{\overline{x^r}}, 0 \le r \le s$, where $0 \le \lambda^0 \le \lambda^1 \le \ldots \le \lambda^s$, with each λ^r the value of the objective function at the iteration *r* and x^r is an optimal solution of the problem **2.3.1** for $\lambda = \lambda^r$. As **2.3.2** is unbounded, after *s* iterations we will have also a growing direction $x^{s+1}, y^{s+1}, v^{s+1}, u^{s+1}$ such that $\begin{bmatrix} x^s \\ y^s \\ v^s \\ u^s \end{bmatrix} + (\lambda - \lambda^s)\begin{bmatrix} x^{s+1} \\ y^{s+1} \\ v^{s+1} \\ u^{s+1} \end{bmatrix}$ is an optimal solution for $\forall \lambda \ge \lambda^s$. So,

$$x^{\lambda}(\lambda) = \begin{cases} \dfrac{\lambda^{r+1} - \lambda}{\lambda^{r+1} - \lambda^r} x^r + \dfrac{\lambda - \lambda^r}{\lambda^{r+1} - \lambda^r} x^{r+1} & for \quad \begin{cases} \lambda^r \le \lambda^{r+1} \\ 0 \le r \le s-1 \end{cases} \\ x^s + (\lambda - \lambda^s) x^{s+1} & for \quad \lambda \ge \lambda^s \end{cases}$$

is an optimal solution of **2.3.1** for any value of $\lambda \ge 0$.

2.4. Applications of the Quadratic Model

We shown a quadratic approach to the portfolio optimization problem and a method to solve the quadratic problem obtaining a portfolio of minimum risk for any feasible level of expected return, parameterizing by the scalar λ. This set of portfolios is called Efficient Frontier and to draw the Efficient Frontier is one of the main applications of the Wolfe’s Method. Now we will show another applications of this quadratic approach.
Having the Efficient Frontier, we can suggest a portfolio for any investor if we have also another function, based in features of the investor, that quantify, for each investiment, the level of satisfaction that he has doing that investment. This function is called *investor’s utility function* and its parameters can be the expected return, its uncertainty quantified by the risk or even both. The expression of an utility function can also change, according to the case. A more technical description of this function is out of the scope of this work and can be seen in [1], [4] and [5]. The investor now wants to find the point of the efficient frontier that maximizes the utility function.

Let U(r) be the utility function expressed in terms of the expected return (it will change from investor to investor), r^l the minimum return of the efficient frontier ($x\'r$ for the solution of **2.3.1** with λ = 0) and r^u the maximum return of the frontier ($x\'r$ for the solution of **2.3.1** with $\lambda \rightarrow +\infty$). If we know U(r) for the investor, we may propose the following problem:

2.4.1 max U(r)
s.t. $r^l \leq r \leq r^u$

If U(r) were a quadratic function (and really may be when we study risk averse investors) we could solve **2.4.1** by the Wolfe's Method.

A different approach for the utility function was proposed by Markowitz in [6]. Markowitz used a quadratic approach of utility function to obtain a set of optimal portfolios in the space *expected return* x *risk*. The approach is based in the Taylor series expansion of U in the portfolio expected return. Therefore, for a risk averse investor:

$$U(R_p) \cong U(x\'r) + U(x\'r)(R_p - x\'r) + \frac{1}{2}U'(x\'r)(R_p - x\'r)^2$$

where $x\'r$ is the portfolio expected return ($E(R_p)$) and R_p is the actual portfolio return.

Applying the expectance operator in the equation above, we will obtain the following identity:

$$E\left[U\left(R_p\right)\right] \cong U(x'r) + \frac{1}{2}U''(x'r)\sigma_p^2$$

This identity looks like the objective function proposed before. Actually it's the same. The expected utility of portfolio was approached by a function whose parameters were only the expected return and the portfolio variance. Therefore a portfolio obtained by maximization of a function $\lambda x'r - \sigma_p^2$ yields a good approach for the maximum expected utility for all the values of $\lambda \geq 0$.

One more application of a quadratic programming model is a very known problem in the financial field. It is related to mutual funds and more details can be seen in [9]. The problem is very seemed with the previous one. Known as Style Analysis, now we want to find a combination of investments in different asset classes (the investment styles), that represents what was the style really followed by the fund manager. Its clear that these classes must be mutually exclusive (or to be well near to this) and must represent all the possible styles to be followed in the market. The objective of the Style Analysis is to find something like coefficients of a linear regression between the fund returns and the classes returns, where each coefficient is the percentage of the total investments of the fund placed in assets of the corresponding class.

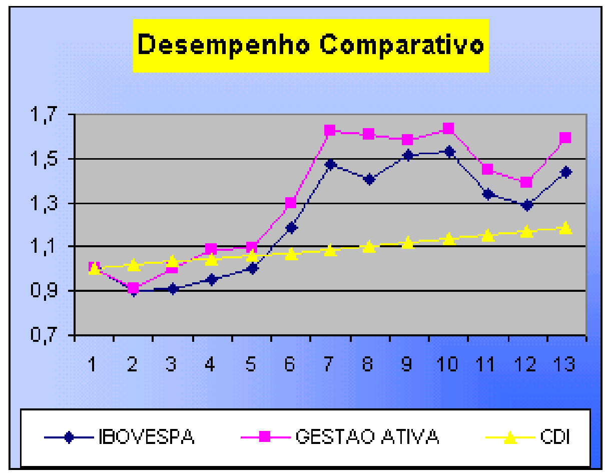
The differences between the Style Analysis and a common regression inhabit in two facts. The first one is that we have constraints. One is a linear constraint, that is to make with that the sum of the coefficients is equal to 1, as in the model of portfolio optimization, because by assumption the investor places all its available money in the portfolio. Another restriction is the signal of the coefficients (they must be positive), since it's practically impossible to operate in a sold position for all the assets of one determined investment class. The second fact is that we do not desire to minimize the sum of the squared errors, but the variance of the errors, therefore the objective of this analysis is to infer the maximum on the fund exposition to variations in the returns of the asset classes during the studied period. It is clearly that, in this point, an algorithm of quadratic programming is necessary, since we desire to minimize a quadratic and convex function (the variance of a sum) with linear constraints. Thus we have one more application of the Wolfe's method.

2.5. A Case Study: Are Efficient Portfolios Really Efficient?

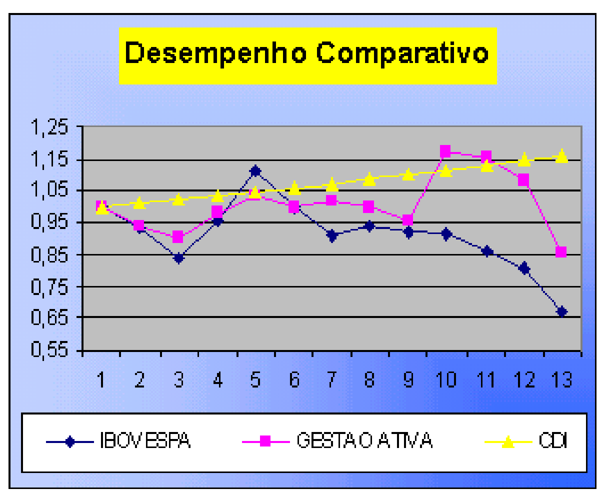
Now we will apply all the presented theory to real data of the Brazilian stock market. We implemented the algorithm in Visual Basic 5.0 (a didactic version is available in http://www.ime.usp.br/~fdias/marreqo.zip) and used the database of the software HSS Stock Market - version 1.3. The price series of the assets were free of dividends. The input data for the algorithm had been estimated as the following. To get consistent results, we decided to make 24 simulations. The expected market portfolio for one month was used as basic parameter. The market portfolio is the portfolio of the efficient frontier that has the greater value for the quotient defined in **1.1.1.** and can be obtained finding the tangency point between the chart of the efficient frontier and a straight line that passes in the corresponding point to the risk free asset. The simulations had been divided in two groups of 12 months, where the first group corresponded to the period of July of 1999 the June of 2000, in which the Brazilian stock market was a bull market, and the second group corresponded to the period of October of 2000 the September of 2001, in which the stock market was evidently a bear market. To generate the suggested portfolio for one determined month, we used monthly passed data in a horizon of twelve months and any costs of transaction and/or tributes had not been considered. We also admitted valid to buy stocks in any amount even fractions.

For each estimate, we selected the 70 stocks that had the greater daily average business volume and that had been negotiated in at least 200 days in the considered horizon.

To estimate the expected return of each asset, first we calculated its geometric average of the simple tax of return, defined as the quotient between the asset price at the time *t*+1 and its price at time *t*. This average tax was composite with a tax equivalent to the monthly dividend yield during the twelve used months. The value obtained minus one is the expected return. To estimate the covariance matrix, we used for each asset the excessive returns from the risk free asset. The risk free asset was the CDI-Over divulged by the CETIP.
The behavior of each attained portfolio was compared with the performance of the market index, considered for these studies the IBOVESPA. Six comparison criteria between the portfolios and the market had been used. The first ones had been the return and the risk, defined as the standard deviation of the excessive returns. To be more criterious in the evaluation, we introduce four other comparison criteria, being the one of them beta of the portfolio, i.e. how much is expected for the portfolio oscillates for each unity oscillation of the market return. It is trivial that the beta of the IBOVESPA will always have to be equal to 1. The beta can be used as a measure of exposure of the portfolio to movements of the market. Thus the ideal strategy of active management will have to get portfolios of low beta during the periods where stock market were bear and of high beta for the periods where the stock market were bull.
Another comparison criteria used matching was the Sharpe Index. This index corresponds to the quotient between excessive return and risk, as defined in **1.1.1**. It can be interesting to classify two portfolios for way of this index, assuming that the higher its value the better is the portfolio, however this index alone can be applied just when the portfolio excessive return is greater than zero, if we do not want to obtain inconsistencies. As a comparison criteria, we used also the Treynor Index. This index corresponds to the quotient between excessive return and beta. As well as in the Index of Sharpe, it can be interesting to classify two portfolios by of this index, assuming that the higher its value the better is the portfolio, however this index alone if applies the portfolios with excessive return (or beta) greater than zero, or else this calculation also can generate inconsistencies. The last comparison criteria between portfolios was the RAP (Risk Adjusted Performance) of Modigliani, that corresponds how much it would be the return of a portfolio if its risk was justifd to the risk of the market index. It is trivial that the RAP of the market index is its proper return.
Tables 1 and 2 show which were the composition of the portfolios suggested for the method for each one of the twelve months of high and for each one of the twelve months of low. Each column corresponds to the portfolio chosen for its respective month. In tables 3 and 4 the monthly returns of each investment alternative are detailed in the first twelve rows and in the rows below the performance indicators of the same ones, being each column corresponding to an investment alternative. In figures 1 and 2 we have a comparative chart of returns between the available options of investment, correspondent how much it would have varied one Brazilian Real invested in each one of them. We can clarely see with that a management strategy based on the Markowitz criterion, optimized for the Method of Wolfe, had a better performance than the market, in the bull periods and in the bear periods.



In both periods we can notice that the portfolios generated for the algorithm had had an average return, Sharpe Index and consistently Treynor Index better than the market. Moreover, even so in both the periods the risk of the portfolio generated for the algorithm was slightly higher (from 3rd decimal place), the performance adjusted to the risk was higher than the market. Moreover, it was noticed that the beta of the generated strategy was higher in the bull period, having fallen in the bear period, what it confirms the previously said about an excellent exposure market risks.



We can notice that exactly during the studied bear period, while the stock market accumulated losses of 19.39% in the 11 first months, our active management accumulated 8.01% profits. These profits had been wasted in the month of September of 2001, that he was catastrophic to the market due the terrorist attacks to the United States, but still thus the proposed strategy won the market in 18.84% in one year. During the high period of the strategy proposal it won the market in 14.90% in one year. Without no shade of doubt, these results are not worthless.

	jul/99	**ago/99**	**sep/99**	**oct/99**	**nov/99**	**dec/99**	**jan/00**	**feb/00**	**mar/00**	**apr/00**	**may/00**	**jun/00**
BBAS4							0,13404	0,08738				
BOBR4					0,05715	0,1105						
BRHA4							0,20916	0,36208		0,23971		
CBEE3												0,09247
CEEB3			0,25598	0,27757								
COGU4							0,02181					
CPSL3	0,07183											
CRUZ3	0,27161	0,34866	0,3828	0,24774		0,03473						
CSNA3	0,09125	0,09003	0,10002	0,05021	0,06287							
ELAT3			0,01984									
ELET3			0,00325				0,01785					
EMBR3						0,11729						
EMBR4							0,14468	0,0941	0,18832	0,09205	0,21945	0,17194
GETI4												0,0606
ICPI4									0,11341			
ITAU4					0,14382	0,08987						
KLAB4				0,07704	0,10354	0,00486						
LAME4	0,10817											
LIGH3											0,17251	0,00501
OSAO4	0,22744											
PCAR4				0,0396	0,08021		0,19202	0,00085				
PRGA4	0,12749	0,18394	0,10813	0,13117	0,14796	0,12985	0,03756					
PTIP4									0,07264	0,23124	0,07052	0,0765
SOES4	0,10221			0,04662	0,34148	0,40472	0,08734					
TCOC3											0,31618	0,32419
TCSP3				0,03922	0,00642	0,0368	0,11812	0,07081	0,05828			
TEPR4			0,12999	0,09084	0,05655				0,17461			
TLPP3												0,04411
TMCP4		0,15365						0,1955	0,22325	0,27951	0,00062	0,00016
TNLP3		0,22373										
TSEP3						0,07139						
USIM5							0,01106	0,1032	0,13688	0,12237		
VALE3								0,08607			0,22073	0,22502
VALE5							0,02638		0,03261	0,03513		

Table 1 – Composition of the efficient portfolios obtained for the period from jul/1999 to jun/2000. Blank spaces means null investment.

	oct/00	nov/00	dec/00	jan/01	feb/01	mar/01	apr/01	may/01	jun/01	jul/01	ago/01	sep/01
AMBV3										0,16285	0,11028	0,18488
AMBV4								0,08101	0,06977			
BBDC3				0,07347					0,0467			
BRDT4										0,12314	0,20737	0,04747
CBEE3	0,1567											
CESP4		0,00091	0,05326	0,0515	0,10434	0,10731	0,12942					
CGAS4			0,03965									
CNFB4												0,37036
CPNE5				0,15277								
CPSL3				0,16701	0,07713	0,06523	0,00404					
CRUZ3		0,11165										
CSNA3								0,02734				
CSTB4									0,01859			
EMBR4	0,17267	0,14418	0,1728	0,19399	0,18011	0,1331	0,2242		0,04247	0,05511	0,02427	
GETI4	0,10612	0,11974			0,21436	0,33725	0,35496	0,09513	0,16682	0,05772		0,0933
GRSU3								0,3673	0,26223	0,30835	0,31994	0,14262
ITAU3					0,24973	0,09895	0,16664	0,2439	0,26451	0,25509	0,32231	0,07218
KLAB4	0,12674	0,24659	0,29391									
LIGHT3	0,14427											
PCAR4				0,06465								
PRGA4												0,08919
PTIP4	0,04571											
TCOC3	0,23266	0,29405	0,35836	0,2554	0,16064	0,25815	0,12074					
TERJ4			0,08202									
TMGR6								0,18532	0,12892	0,03775	0,01583	
TSPP3					0,01371							
USIM5	0,01515	0,08288										
VALE3				0,04121								

Table 2 – Composition of the efficient portfolios obtained for the period from oct/2000 to set/2001. Blank spaces means null investment.

	MARKET	PORTFOLIO	CDI
jul/99	-10,193%	-8,988%	1,660%
ago/99	1,178%	9,925%	1,624%
sep/99	5,131%	8,419%	1,399%
oct/99	5,348%	0,839%	1,173%
nov/99	17,761%	18,208%	1,304%
dec/99	24,046%	25,773%	1,580%
jan/00	-4,113%	-1,396%	1,371%
feb/00	7,762%	-1,465%	1,440%
mar/00	0,906%	3,284%	1,440%
apr/00	-12,811%	-11,507%	1,293%
may/00	-3,739%	-3,773%	1,486%
jun/00	11,841%	14,262%	1,387%
ACUM.	43,876%	58,777%	18,573%

MEAN RET.	3,593%	4,465%	1,430%
MEAN EXC.	2,163%	3,035%	0,000%
RISK	10,397%	10,681%	0,000%
BETA	1,000000	0,8975341	
SHARPE	0,208044	0,2841785	
TREYNOR	2,163%	3,382%	
RAP	3,593%	4,259%	

Table 3 – Comparison between jul/99 and jun/00

	MARKET	PORTFOLIO	CDI
oct/00	-6,661%	-6,309%	1,280%
nov/00	-10,628%	-3,606%	1,216%
dec/00	14,842%	8,617%	1,135%
jan/01	15,814%	5,572%	1,203%
feb/01	-10,078%	-3,417%	1,009%
mar/01	-9,144%	1,719%	1,248%
apr/01	3,318%	-1,971%	1,179%
may/01	-1,797%	-3,803%	1,271%
jun/01	-0,614%	22,386%	1,269%
jul/01	-5,529%	-1,843%	1,431%
Ago/01	-6,645%	-6,291%	1,600%
Sep/01	-17,173%	-20,739%	1,323%
ACUM.	-33,231%	-14,393%	16,262%

MEAN RET.	-2,858%	-0,807%	1,264%
MEAN EXC.	-4,122%	-2,071%	0,000%
RISK	9,608%	9,856%	0,000%
BETA	1,00000	0,59240	
SHARPE	-0,42898	-0,21011	
TREYNOR	-4,122%	-3,495%	
RAP	-2,858%	-2,124%	

Tabela 4 – Comparison between oct/00 and sep/01

2.6. Limitations of the Quadratic Model

The quadratic approach of the expansion of the utility function in power series turns the model most easy of being applied, however it is more limited, since it is not considering central moments of higher order, as the skewness and kurtosis. This can be supported by the hypothesis of that the returns of the market are i.i.d., and as a consequence of the central limit theorem, it would generate a normal distribution for the returns of an risky asset. Since the returns of asset with risk have normal distribution, its skewness is null and its kurtosis excess, what becomes useless the study of central moments of higher order. However it is a well known fact that the distribution of the returns of an risky asset does not follow actually a normal distribution, and do not consider these higher order moments does not give a good approach for the utility.

An example can be formulated in the following way: after market studies, one conclude that the expected return of an asset is of 10% and its risk is of 3.5%. This would mean to say that, assuming normality in the returns, in approximately 67% of the cases the return of this asset would be between 6.5% and 13.5%. However if this distribution were not normal, having one determined level of skewness, in fact the used reliable interval in the previous inference would not mean more 67% of the cases.

In particular, if the asymmetry is positive, a reliable interval that starts from point 6,5% and includes same 67% of the cases will contain higher values than 13,5% being, for example, the interval between 6.5% and 14.5% as illustrated in Figure 4. More catastrophic will be if the skewness if not taken into account but it exists and either negative. In this situation, the hypothetical interval of 67% of the cases from 6.5% could be between 6.5% and 12%, as in figure 5, what it makes our forecasts that seemed to be logical in the truth to be excessively optimistical. The higher is, in module, the value of the skewness coefficient of the distribution of returns, weakker will be the concept of variance and covariance as a measure of risk of an investment. Intuitively, we conclude that a good approach of the function utility must take into account the skewness and to try maximizes it. Moreover, it has studies that they affirm that even when the assets possess normal distribution, dynamic strategies of purchase and sale can generate significantly skewed distributions. In this work we only speak of basic assets, but it can be desired to include derivatives in the way active them with risk with which we desire to work. In this point, it is well known that derivatives follow a highly skewed distribution of returns, since the price of a derivative in the expiration date is explained by the combination between a constant function equal to zero for values below of a point and linear for values above of this point. Thus, the quadratic model becomes impracticable for the inclusion of derivatives in the portfolio optimization model.

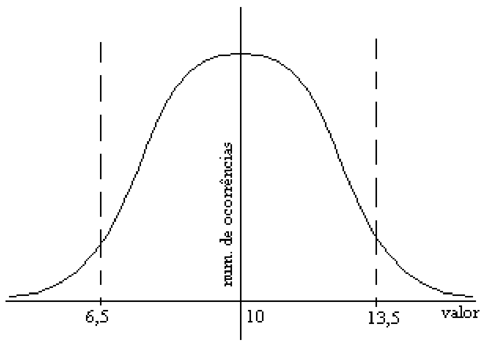


Fig. 3 – Normal probability distribution with average 10 and standard deviation 3,5.

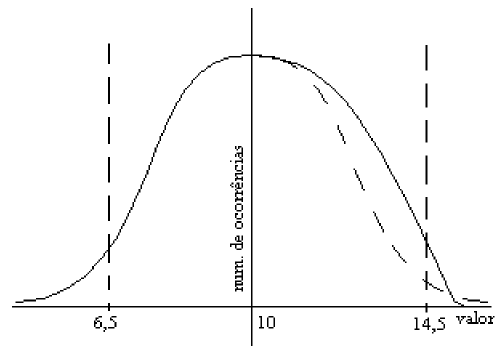


Fig. 4 – Probability distribution with positive skewness, average 10 and standard deviation 3,5. The dotted line is a normal distribution of same average and standard deviation.

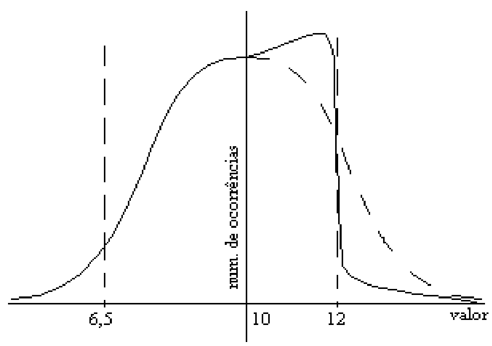


Fig. 5 – Probability distribution with negative skewness, average 10 and standard deviation 3,5. The dotted line is a normal distribution of same average and standard deviation.

Moreover, the optimization model of considered above yieds a solution for just one period, without taking into consideration expected utilities for posterior periods. Currently stochastic programming algorithms are also subject of other research that do not consider more the objective function of the problem to be decided as a function of one alone period, but a function whose variables are determined by stochastic processes and we now desire to maximize the utility for the following period added to the expected utility for the other diverse periods. This expected utility could be estimated by a tree that describes all the possible scenes to occur later, also containing the probability of this occurrence. More advanced studies in stochastic programming are out of the scope of this work.

Another limitation of the model studied so far is that linear constraints have little power of nonlinear constraints. Nonlinear models are more general and allowing constraints on nonlinear factors could enrich still more the model and leaves it closer to the reality.

Finally, there are studies that deny at all the hypothesis that the market is normal or even near to it and go to of nonlinear dynamic models with long memory and extreme values. Mandelbrot considered in [12] a form to analyze the return series as a fractal distribution, oposing totally the approach of risk by a covariance matrix. Tonis Vaga considered in [11] the Hypothesis of the Coherent Market, where the probabilist distribution of the returns changes dinamically with the time based on a function that also was used by Ising to model the ferromagnetism. These are totally different approaches of the market, but not the less promising ones.

3 Portfolio Skewness

3.1. The Model Formulation

We had shown an investor that wants to maximize the expected value of one utility function *U*(*R*), where *R* is the expected return of its investment. The attained results are based in the Taylor series expansion of this function until its second derivative, generating as objective function when we use the expectance operator something whose parameters are the two first central moments of the portfolio return.

Now we will extend the attained result, considering the third central moment – the skewness of the returns in relation of the mean.

For a portfolio with *n* risky assets, the first three central moments of the portfolio return (R_p) are given by:

$$\overline{R}_p = E[R_p] = \sum_{i=1}^{n} x_i \overline{R}_i$$

$$\sigma_p^2 = E\left[\left(R_p - \overline{R}_p\right)^2\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij}$$

$$m_p^3 = E\left[\left(R_p - \overline{R}_p\right)^3\right] = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} x_i x_j x_k m_{ijk}$$

with $\overline{R}_i$ = expected return of the asset *i*

σ_{ij} = covariance between the returns of *i* and *j* (for *i* = *j* it is the variance)

m_{ijk} = co-skewness between the returns of *i*, *j* and *k*, defined by the third central moment not normalized.

So for a risk averse investor, the portfolio expected utility will be:

$$E\left[U(R_p)\right] \cong E\left[U(\overline{R}_p)\right] + \frac{U''(\overline{R}_p)E\left[(R_p - \overline{R}_p)^2\right]}{2!} + \frac{U'''(\overline{R}_p)E\left[(R_p - \overline{R}_p)^3\right]}{3!} \cong U(R_p) + \frac{U''(\overline{R}_p)\sigma_p^2}{2} + \frac{U'''(\overline{R}_p)m_p^3}{6}$$

In this way, maximize the expected utility for a risk averse investor is to solve:

3.1.1

$$\min \quad \lambda x'r - x'Cov.x + \alpha \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} x_i x_j x_k m_{ijk}$$

$$s.a. \quad x \in \Im, \Im = \{x \mid Ax \le b, l \le x \le u\}$$

where α. is a parameter to the second degree of freedom of this new problem, the investor skewness preference.

Note that varying at the same time the parameters os parâmetros. λ and α, we will obtain an efficient surface in the 3-dimensional space *risk* x *expected return* x *skewness*.

3.2. Conclusions and Extensions for Future Research

We shown a brief description of the Markowitz' quadratic model. This model shown to explain very well the market behaviour. However we saw that a quadratic approach is good only when the market follows a normal distribution. There are evidences that the market follows a dynamical behaviour, being normal sometimes and do not so in other periods. Therefore we proposed another model where the third central moment is introduced and the investor skewness preference is taked into account.

At this time, new doubts appear:

- What method is more efficient to solve it? Would be the gradient method?
- Methodology for the attainment of the efficient surface: is it really necessary to solve for all the parameters or is possible to model the optimality conditions of this problem as a new parametric linear or quadratic problem, and to get critical values of the parameter and therefore all the surface?
- Although the problem has two degrees of freedom, is there some relation between them? Is possible to model the problem using just one parameter?

These and others doubts that probably will appear are a broad research field to be explored. To explain them is a difficult task, but when concluded will generate a great contribution to the current theory, mainly when related to efficient portfolios including options, since the quadratic model does not give any support to this case.

Not only this, a large research field is opened for applications of models highly nonlinear in the financial market, like Fractals, Chaos Theory and Extreme Value Theory.

Much work still remains to be done.

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